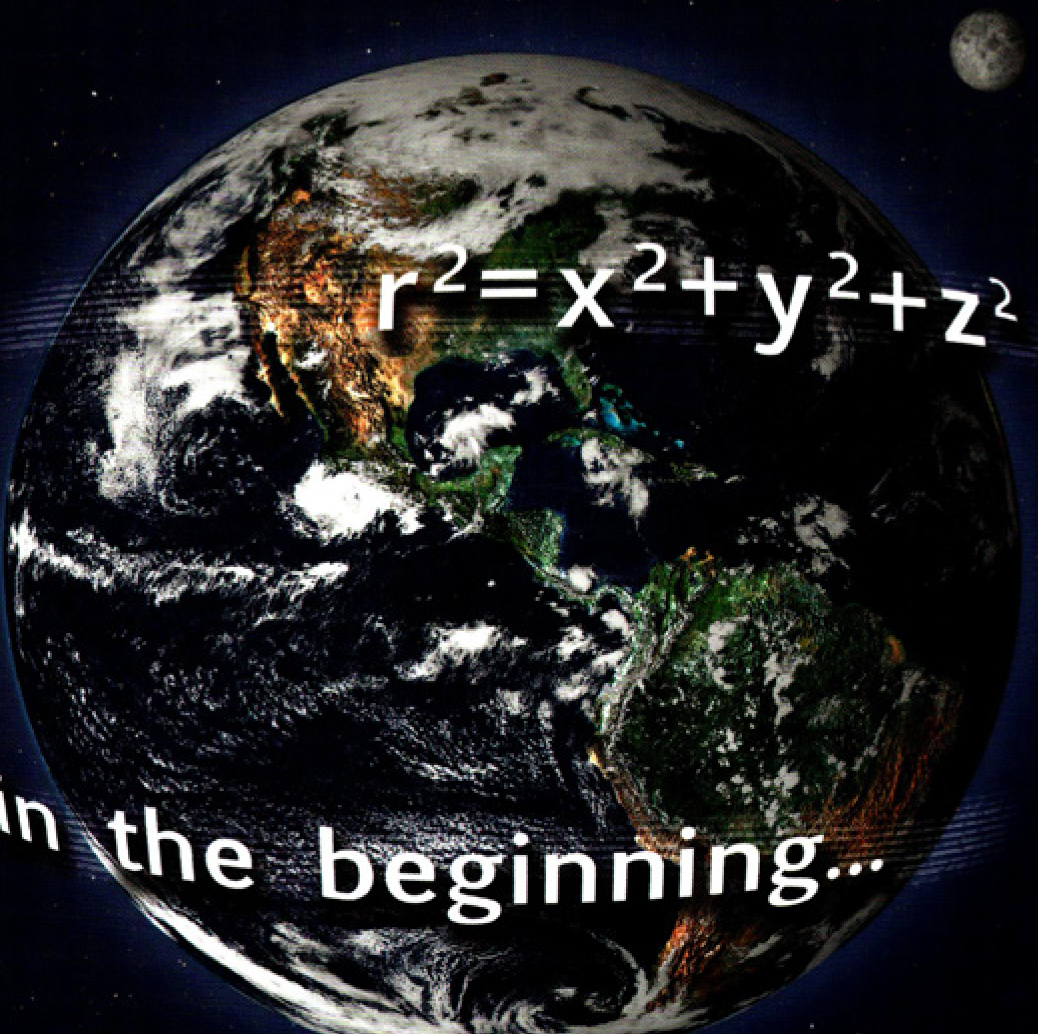
ELEMENTARY ALGEBRA

HAROLD R. JACOBS



REVISED EDITION

ELEMENTARY
ALGEBRA


An' the formoola we uses is algebra---invented by a fella name of "Al."
Inasmuch as he done such a good job, the formoola was named after him---bein' called Al-gebra 'cause he was a zebra.
Al-Zebra?!
Zebra don't got no "G" in it. So how could---
Who said it do?
An' algebra don't got no "Z."
Algebra got nothin' but "Z's" an' "X's" an' stuff in it!
True, true.

ELEMENTARY ALGEBRA

Harold R. Jacobs

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Supplements to Elementary Algebra

The *Teacher's Guide* which includes a convenient daily schedule, Set III exercises for additional practice, and a complete set of tests, including chapter tests, mid-year and final exams and answer keys.

The *Solutions Manual* with complete solutions to all of the Set I–IV exercises.

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Foreword

I am just an algebra teacher, more interested in how students learn, and how to open doors of opportunity for people than I am a mathematician. Harold Jacobs, through his texts, has taught me something about how to be a mathematician.

You will find that this book is far more than a text. It is unusual in that it has withstood the test of time, not because it is a reference for the subject, but by the popular demand of people who want these things from any text: the thoughtful, meticulous exposition of a subject as well as a genuine love for the subject, the understanding of the student, and the appreciation for the instructor. It also helps if the author has fun doing it and can invite the student along for the ride.

Every time I open this book to teach a new lesson I gain another insight, another 'Aha!' moment. I am a perpetual student of Harold Jacobs, even though I was never in his classroom. Harold Jacobs's books have that effect on people.

I am grateful to God that the book will continue to be published! It is a strong tool in the hands of just an adequate instructor like myself, attempting to crank open brains and doors of opportunity.

Molly Crocker
Adequate Instructor and Independent Educator
October 2015

Photograph by Roy Bishop

A Letter to the Student

The English philosopher and scientist Roger Bacon once wrote: "Mathematics is the gate and key of the sciences. . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world."

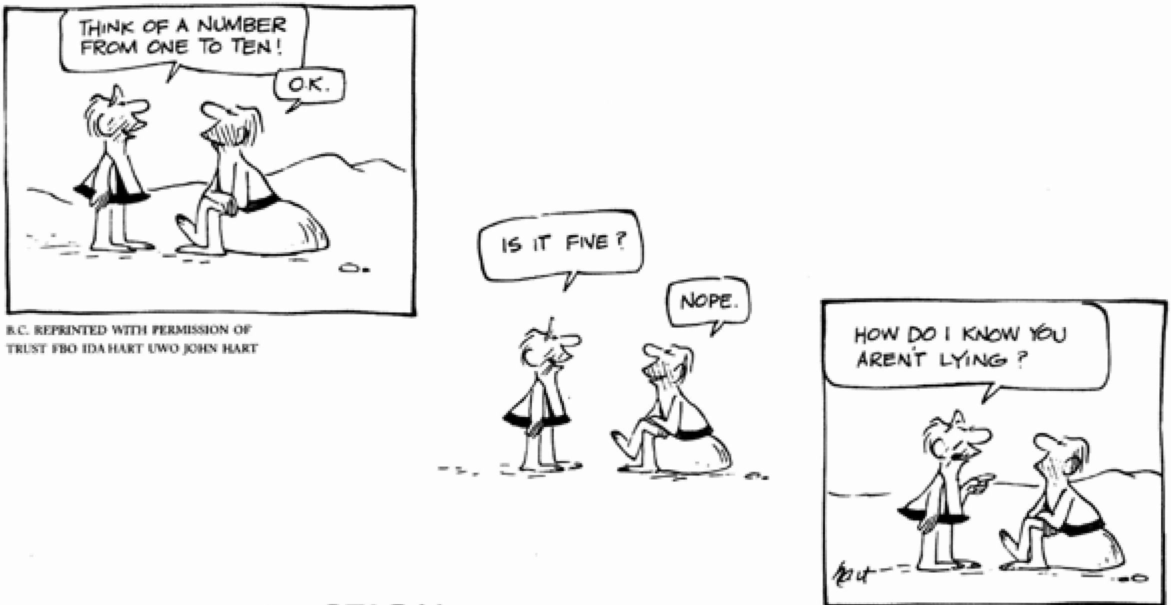
In turn, algebra is the gate and key of mathematics. For this reason, colleges and universities require mastery of algebra in preparation for studying not only the sciences, but also such subjects as engineering, medicine, architecture, philosophy, psychology, and law.

Although many problems that can be solved by algebra can also be worked out by common sense, their translation into algebraic form generally makes them easier to deal with. Because of this, algebra has become the language of science. The goal of this course is to learn how to use this language.

Success in algebra depends on a combination of talent and effort. A few people are so gifted in mathematics that they can succeed with very little effort. For most people, however, diligent practice is the key to success. Like developing ability in a sport, becoming good at algebra takes practice. It is my hope that this book will help you both to enjoy the subject and to be successful in your studies.

Harold R. Jacobs

ELEMENTARY ALGEBRA



INTRODUCTION
A Number Trick

Think of a number from one to ten. Add seven to it. Multiply the result by two. Subtract four. Divide by two. Subtract the number that you first thought of. Is your answer five?

Number tricks such as this have long been popular. That the final result can be known by someone who doesn't know which number was originally chosen is surprising.

How does the trick work? If we make a table (like the one at the top of the next page) showing what happens when it is done with each number from one to ten, some patterns appear.

Would these patterns continue if the table were extended to include other numbers? If we began by thinking of eleven, would the answer at the end still be five? What if we began with one hundred? Would we get five at the end if we began with zero? Do you think it is correct to assume that the trick will work for *any* number you might think of?

Even though you may feel that the answer to every one of these questions is yes, *how* the trick works is still not clear. Merely doing arithmetic with a series of different numbers cannot reveal the secret of why they all lead to the same result.

The number thought of:	1	2	3	4	5	6	7	8	9	10
Add seven:	8	9	10	11	12	13	14	15	16	17
Multiply by two:	16	18	20	22	24	26	28	30	32	34
Subtract four:	12	14	16	18	20	22	24	26	28	30
Divide by two:	6	7	8	9	10	11	12	13	14	15
Subtract the number first thought of:	5	5	5	5	5	5	5	5	5	5

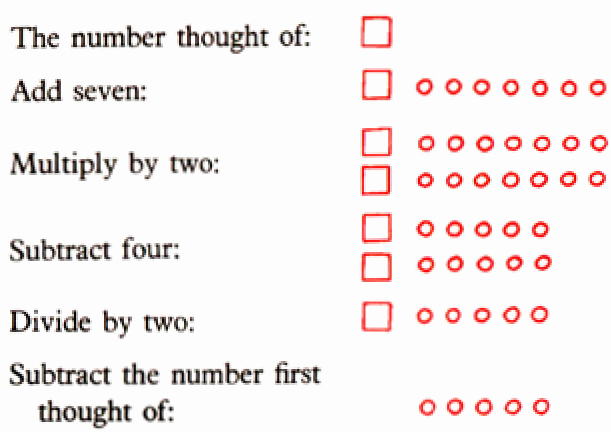
There is a simple way, however, to discover the secret. Instead of writing down a specific number at the start, we will use a symbol to represent whatever number might be chosen. We will begin with a box.

Throughout the trick this box will represent the number originally chosen.

The next step in the trick is to add seven. We will represent numbers we know with sets of circles, and so seven will look like this:

To show the result of adding seven to the number, we draw seven circles beside the box.

If we illustrate the entire trick in this way, it looks like this:



The pictures make it easy to see why, no matter what number we start with, the answer at the end of the trick is always five. The box representing the original number disappears in the last step, leaving five circles.

Doing arithmetic with symbols rather than specific numbers is the basis of algebra. The explanation with the boxes and circles of what is happening throughout the number trick is an example of this. One of our goals in learning algebra will be to learn how to set up and solve problems using symbols such as these.

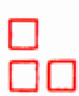
Exercises

1. Here are directions for another number trick and part of a table to show what happens when the trick is done with each number from one to five.

Think of a number:	1	2	3	4	5
Double it:	2	4			
Add six:	8				
Divide by two:	4				
Subtract the number that you first thought of:	3				

   a) Copy and complete the table.
   b) Does your table prove that the trick will work for *any* number?
   c) Show how the trick works by illustrating the steps with boxes and circles. The first two steps are shown below.

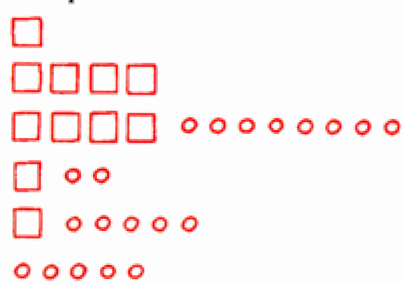
   Think of a number:
   Double it:



   d) Do your drawings prove that the trick will work for *any* number?

2. The pictures below illustrate the steps of another number trick. Tell what is happening in each step in words.

Step 1.
Step 2.
Step 3.
Step 4.
Step 5.
Step 6.



3. In the next number trick, we will study the effect of changing some of the directions.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by two.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

   a) What is the result at the end of this trick?

b) Suppose that the second step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add six.
Step 3.	Multiply by two.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

The trick will still work, even though the result at the end is changed. How is it changed?

c) Suppose instead that the fourth step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by two.
Step 4.	Subtract six.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

d) Suppose instead that the third step were changed as shown below.

Step 1.	Think of a number.
Step 2.	Add four.
Step 3.	Multiply by four.
Step 4.	Subtract four.
Step 5.	Divide by two.
Step 6.	Subtract the number that you first thought of.

What effect does this have on the trick?

4. Here is the beginning of a number trick. Can you make up more steps so that it will give the same answer for any number a person might choose?

Think of a number.
Triple it.
Add twelve.

Chapter 1

FUNDAMENTAL OPERATIONS

"Can you do Addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice. "I lost count."

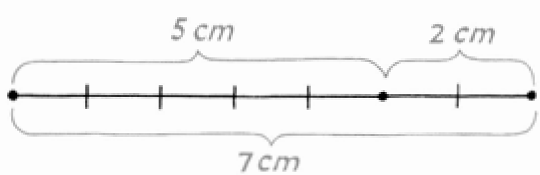
LEWIS CARROLL, *Through The Looking Glass*

LESSON 1
Addition

Soon after a child is able to count, he learns how to add. The two operations are closely connected, as anyone who has ever added by counting on his fingers knows. Consider the problem of adding the numbers represented by these two sets of circles:

At first a child finds the answer by counting all of the circles. Then he learns the fact that $5 + 2 = 7$.

Another way to picture addition is by lengths along a line. This figure also illustrates the fact that $5 + 2 = 7$.



The result of adding two or more numbers, called their **sum,** does not depend on either the order of the numbers or the order in which they are added. To find

the number of circles in the pattern above, for example, we could add the numbers of circles in the four rows from top to bottom:

$$1 + 2 + 3 + 4$$

or from bottom to top:

$$4 + 3 + 2 + 1$$

Either way, we get the same number: 10.

In algebra, it is often necessary to indicate the sum of two or more numbers without actually being able to add them. For example, in illustrating the number trick that appears in the introduction to this book, we used a box to represent the original number and a set of circles to represent the number seven:

Original number Seven

To represent their sum, we drew the seven circles beside the box:

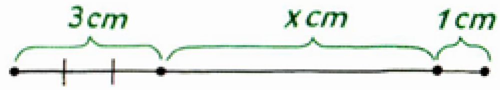
The sum of the original number and seven

Instead of bothering to draw pictures like this, it is easier to represent the original number with a letter, such as x, and simply write

$$x + 7$$

The expression $x + 7$ means "the sum of x and 7." If we replace x with 1, $x + 7 = 1 + 7 = 8$. If we replace x with 2, $x + 7 = 2 + 7 = 9$, and so forth. Because x can be replaced by various numbers, it is called a **variable**.

If we know both numbers being added, such as 4 and 5, we can write their sum as a number, 9. If we know only one number or neither one, the best that we can do is to write an expression such as $x + 2$ or $x + y$. The length of the line segment below, for example, is the sum of the lengths of the three marked segments.



To indicate this sum, we can write $3 + x + 1$ or, more briefly, $x + 4$. Without knowing the length labeled x, we cannot simplify this answer any further.

Exercises

TEACHER: Haven't you finished adding up those numbers yet?
STUDENT: Oh, yes. I've added them up ten times already.
TEACHER: Excellent! I like a student who is thorough.
STUDENT: Thank you. Here are the ten answers.*

Set I

Find each of the following sums.

1. $1000 + 700 + 70 + 6$
2. $999 + 99 + 9$
3. $1 + 0.9 + 0.08 + 0.004$
4. $20 + 0.2 + 0.002$
5. $1 + 12 + 123 + 1234$
6. $1111 + 222 + 33 + 4$
7. $1 + 1.2 + 1.23 + 1.234$
8. $1.111 + 2.22 + 3.3 + 4$
9. $0.7 + 0.70 + 0.700 + 0.7000$
10. $0.5 + 0.55 + 0.555$

Set II

11. Write a number or expression for each of the following.

a) The sum of 10 and 7.
b) The sum of x and 7.
c) The sum of 10 and y.
d) The sum of x and y.
e) Four added to 8.
f) Four added to z.
g) The sum of 2, 5, and 1.
h) The sum of x, 5, and 1.
i) The sum of 2, y, and 1.
j) The sum of x, y, and 1.

* Alan Wayne, in *Mathematical Circles Revisited* by Howard W. Eves. © Copyright Prindle, Weber & Schmidt, Inc. 1971.

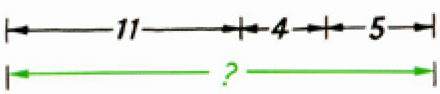
12. In the figures below, the box represents any number and the sets of circles represent specific numbers.

Figure 1

Figure 2

a) What addition problem is illustrated by Figure 1?
b) What is the answer to the problem?
c) Write an algebraic expression to represent the addition problem illustrated by Figure 2.
d) What is the answer to the problem if the box represents 2?
e) What is the answer to the problem if the box represents 4?

13. What is the length marked with a question mark in each of these figures?

a)


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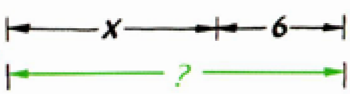

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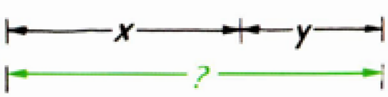

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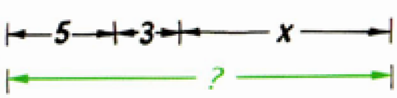
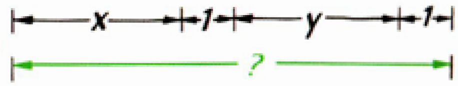
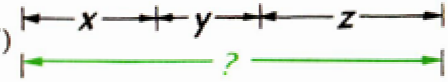


e)


f)


14. The figure below can be used to show that $3 + 7$ and $7 + 3$ are the same number, depending on whether the figure is read from left to right or from right to left.

Draw boxes and circles to show that
a) $x + 6$ and $6 + x$ mean the same thing.
b) $2 + x + 5$ and $x + 7$ mean the same thing.
c) $x + 4 + x$ and $4 + x + x$ mean the same thing.

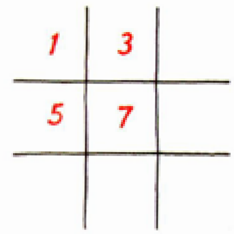
15. The expression $x + y + 2$ represents the sum of x, y, and 2. If x is 1, it can be written as $1 + y + 2$ or $y + 3$. How can $x + y + 2$ be written if
a) x is 8?
b) x is 9?
c) y is 3?
d) y is 0?
e) x is 6 and y is 2?

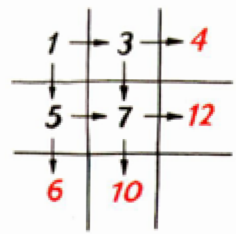
16. Mr. Benny is 39 years old.
a) How old will he be in 5 years?
b) How old will he be in x years?
c) How old will he be 6 years after that?
Mrs. Benny is x years old.
d) How old will she be in 5 years?
e) How old will she be in y years?
f) How old will she be z years after that?

Set III

The exercises in the Set III sets are similar to those in Set II. For students who need or would like additional practice, they can be found in the Teacher's Guide for this book.

Set IV A Number Puzzle





Numbers have been written in four spaces in this tic-tac-toe design. If we add across the rows and down the columns, we get the sums shown in the second figure. If we now add across the bottom row and down the last column, the answers are the same number:

$$6 + 10 = 16 \quad \text{and} \quad 4 + 12 = 16$$

Is this just a coincidence or would it happen if we started with *any* set of four numbers?

Draw a tic-tac-toe design and, in the same spaces as those in the example above, write four numbers of your own choosing. Add the rows and columns and see what happens. Can you explain why?

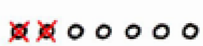
LESSON 2
Subtraction

The month of September 1752 was one of the strangest months in history. The day following September 1 was September 13!

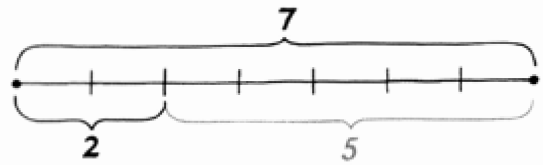
This was done to bring the calendar back into line with the seasons. The calendar established by Julius Caesar in 45 B.C. had as its basis a standard year of 365 days with every fourth year, "leap year," having 366. This resulted in the average length of a year being 365.25 days, whereas the earth in fact travels once around the sun in about 365.24 days. For a short period, this error didn't amount to much, but after many centuries it became so great that it had to be corrected.

The number of days left in the month of September 1752 can be found by subtraction: $30 - 11 = 19$. Subtraction is the opposite of addition because we are "taking away" rather than "adding to." The two operations are closely related, however, because to every subtraction problem there corresponds an addition problem: $30 - 11 = 19$ because $19 + 11 = 30$.

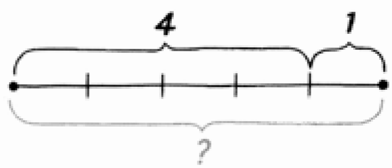
To represent a subtraction problem such as $7 - 2$ by means of circles, we might draw seven circles from which two have been "taken away" by being crossed out.

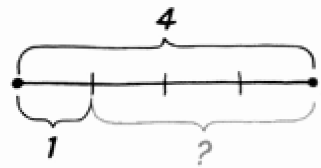


Subtraction can also be pictured by lengths along a line. The figure below is another way of showing that $7 - 2 = 5$.



Although addition and subtraction are closely related, there is an important difference between the two operations. The sum of two numbers does not depend on the order of the numbers. The length marked with a question mark in the figure at the left below can be written either as $4 + 1$ or $1 + 4$.





The result of subtracting one number from another, called their **difference,** does depend on the order of the numbers. The length marked with a question mark in the figure at the right is $4 - 1$, not $1 - 4$. When we refer to the difference between two numbers, we mean the number that results from subtracting the second number from the first.

Exercises

Set I

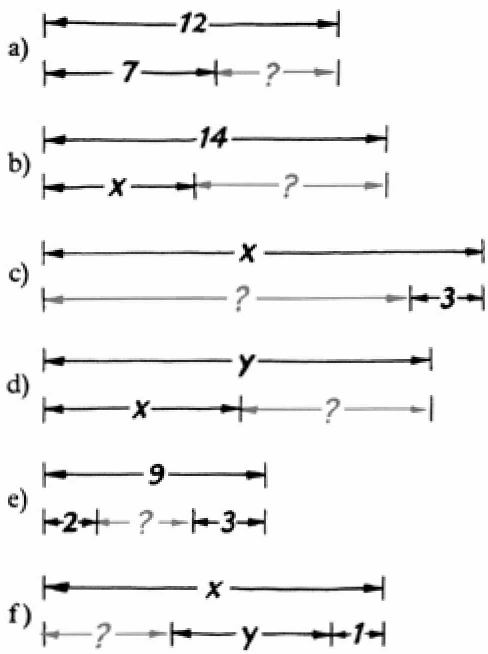
Find each of the following differences.

1. 22222 – 2000
2. 666 – 77
3. 1000 – 123
4. 4.321 – 1
5. 4.321 – 0.1
6. 3.1416 – 3.1416
7. 1 – 0.9
8. 1 – 0.99
9. 1812 – 18.12
10. 181.2 – 1.812

Set II

11. Write a number or expression for each of the following.
    a) The difference between 10 and 7.
    b) Six decreased by x.
    c) Six taken away from x.
    d) Three less than 11.
    e) One less than x.
    f) The difference between x and y.
    g) The result of subtracting x from 4.
    h) Four subtracted from x.

12. What is the length marked with a question mark in each of these figures?



13. Find the value of each of the following expressions for the numbers given.
    a) $x - 4$ if x is 6.
    b) $x - 4$ if x is 7.
    c) $x - 4$ if x is 14.
    d) What happens to the value of $x - 4$ as x gets larger?
    e) $15 - x$ if x is 3.
    f) $15 - x$ if x is 4.
    g) $15 - x$ if x is 10.
    h) What happens to the value of $15 - x$ as x gets larger?

14. Find the value of each of the following for the numbers given.
    The sum of x and $y - 3$
    a) if x is 7 and y is 4.
    b) if x is 2 and y is 11.
    The difference between $x + y$ and 3
    c) if x is 7 and y is 4.
    d) if x is 2 and y is 11.
    e) Can you explain why the answers to parts c and d are the same as those to parts a and b?

15. The sum of the numbers on any two opposite faces of a die is 7. Suppose that a die is thrown.
    a) If the number showing on the top of it is 3, what is the number on the bottom?
    b) If the number showing on the top of it is x, what is the number on the bottom?
    Suppose that two dice are thrown.
    c) If the sum of the two numbers showing on top is 8, what is the sum of the two numbers on the bottom?
    d) If the sum of the two numbers showing on top is y, what is the sum of the two numbers on the bottom?

16. Babar weighs 7,000 pounds.
    a) If he loses x pounds, how much will he weigh?
    b) If he gains y pounds, how much will he weigh?

17. The amount of profit that Shirley Feeney makes selling sandwiches depends on how much they cost her and how much she sells them for.

a) If peanut butter sandwiches cost her 21 cents each and she sells them for 45 cents, how much profit does she make on each one?
b) If jelly sandwiches cost her x cents each and she sells them for y cents, how much profit does she make on each one?
c) If egg sandwiches cost her x cents each and she wants to make a profit of 30 cents, how much should she sell them for?
d) If she sells ham sandwiches for 95 cents each and makes a profit of y cents on each one, how much do they cost her?

Set IV

"Forty-eight, forty-nine, fifty, seventy-five, nine, ten, twenty."

This seems like a strange way to count and yet clerks in stores do it all the time. What is going on? Can you tell what problem is being solved? Is the problem being solved by addition or subtraction?

"Six times six is 54! Don't they teach you anything at that school?"

LESSON 3
Multiplication

Learning the multiplication table is not an easy task. When you first learned how to multiply, you did it by adding. For example, the problem 3×5 can be illustrated by three sets of circles with five circles in each set.

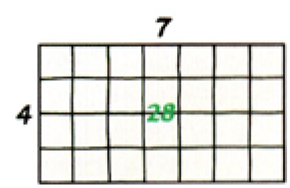
The circles can also be arranged in three rows to form a rectangle.

Both patterns show that $3 \times 5 = 5 + 5 + 5 = 15$. In learning the multiplication table, you memorized the answers to problems such as this so that pictures and adding became unnecessary.

The result of multiplying two or more numbers is called their **product.** Another way to picture a product is by means of area. The rectangle at the right, for example, is divided into 4 rows of squares with 7 squares in each row: it contains

$$4 \times 7$$

squares in all. The area of the rectangle, 28, is the product of its dimensions, 4 and 7.



Something that helps in learning the multiplication table is the fact that if

$$4 \times 7 = 28$$

then it is also true that

$$7 \times 4 = 28$$

The product of two or more numbers, like their sum, does not depend on either their order or the order in which they are multiplied.

Each of the number tricks that we considered in the introductory lesson included a step consisting of multiplication. For example, if we are told to think of a number and multiply it by four, the result might be illustrated by a set of four boxes:

If we use a letter, such as x, to represent the number thought of, we might write:

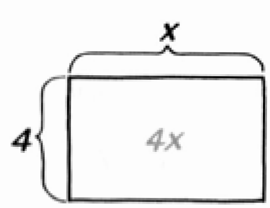
$$4 \times x$$

Because the symbol for multiplication used in arithmetic looks so much like the letter x, however, it is not ordinarily used in algebra. Instead, we simply write $4x$ with the understanding that this means "4 times x." We can't indicate the product of two numbers such as 3 and 5 this way because 35 means "thirty-five," not "three times five." To indicate that the 3 and 5 are two separate numbers, we can either enclose them in parentheses, (3)(5), or insert a raised dot between them, $3 \cdot 5$.

In this lesson we have observed that the product of two numbers, such as $4x$, can be interpreted either as repeated addition,

$$x + x + x + x$$

or as the area of a rectangle whose dimensions are 4 and x.



In the next lesson, we will see how these ideas can be applied to division.

Exercises

Set I

Find each of the following products.

1. $100 \cdot 360$
2. $(5)(142857)$
3. $271 \cdot 287$
4. $(0.05)(20)$
5. $(1.5)(8.23)$
6. $(8.23)(1.5)$
7. $(0.7)(1.1)(1.3)$
8. $(7)(1.1)(1.3)$
9. $(7)(11)(1.3)$
10. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Set II

11. Draw figures as indicated.
    a) A figure with circles to show that $4 \cdot 3$ and $3 \cdot 4$ are the same number.
    b) A figure with boxes to illustrate $5x$ if each box represents x.
    c) A rectangle divided into squares to illustrate $2 \cdot 7$.

12. Write a number or expression for each of the following.
    a) The product of 5 and 6.
    b) The sum of 5 and 6.
    c) The product of 5 and x.
    d) The sum of 5 and x.
    e) The product of x and y.
    f) The sum of x and y.
    g) The product of x and x.
    h) Eight multiplied by x.
    i) Eight subtracted from x.
    j) The sum of 2, 7, and x.
    k) The product of 2, 7, and x.
    l) The sum of 10, y, and 3.
    m) The product of 10, y, and 3.
    n) The sum of 4, x, and y.
    o) The product of 4, x, and y.

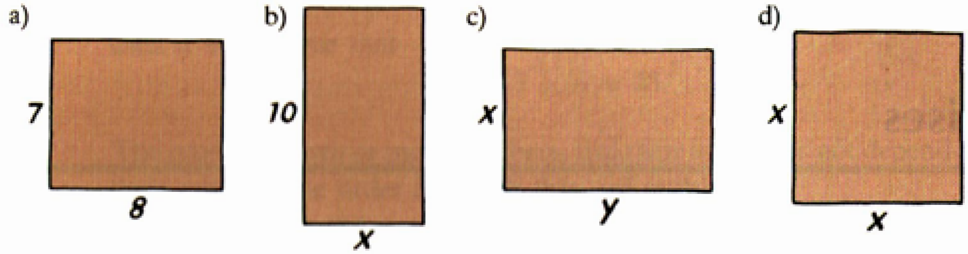
13. The multiplication problem $4 \cdot 3$ and the addition problem $3 + 3 + 3 + 3$ are equivalent. Write a multiplication problem equivalent to each of the following addition problems.

    a) $2 + 2 + 2 + 2 + 2 + 2$
    b) $6 + 6$
    c) $x + x + x + x + x$
    d) $\underbrace{7 + 7 + \cdots + 7}_{\text{11 of them}}$
    e) $\underbrace{7 + 7 + \cdots + 7}_{x \text{ of them}}$
    f) $\underbrace{y + y + \cdots + y}_{x \text{ of them}}$

    Write an addition problem equivalent to each of the following multiplication problems.

    g) $3 \cdot 17$
    h) $4x$
    i) $y \cdot 2$
    j) yz

14. The area of a rectangle is the product of its length and width. What is the area of each of these rectangles?



15. Although the name suggests that they have 100 legs, some centipedes have only 28 legs, whereas others have as many as 354.
   a) How many legs do 5 centipedes have altogether if each one has 28 legs?
   b) How many legs do x centipedes have altogether if each one has 354 legs?

16. Because there are 60 minutes in an hour, there are $60x$ minutes in x hours.
   a) How many days are there in x weeks?
   b) How many hours are there in x days?
   c) How many minutes are there in one day?
   d) How many minutes are there in x days?
   e) How many minutes are there in x weeks?
   f) How many years are there in x centuries?
   g) How many months are there in x centuries?

17. Miss Haversham's Hupmobile gets about 11 miles per gallon.
   a) Approximately how many miles should she be able to travel on a full tank of 15 gallons?
   b) Approximately how many miles can she travel on x gallons of gas?

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